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The Stochastic Heat Equation: Feynman–Kac Formula and Intermittence

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We study, in one space dimension, the heat equation with a random potential that is a white noise in space and time. This equation is a linearized model for the evolution of a scalar field in a space-time-dependent random medium. It has also been related to the distribution of two-dimensional directed polymers in a random environment, to the KPZ model of growing interfaces, and to the Burgers equation with conservative noise. We show how the solution can be expressed via a generalized Feynman–Kac formula. We then investigate the statistical properties: the two-point correlation function is explicitly computed and the intermittence of the solution is proven. This analysis is carried out showing how the statistical moments can be expressed through local times of independent Brownian motions.

KEY WORDS: Stochastic partial differential equations; Feynman–Kac formula; random media; moment Lyapunov exponents; intermittence; local times.

1. INTRODUCTION

We consider the linear stochastic partial differential equation (SPDE)

$$\partial_t \psi_t(x) = \frac{\nu}{2} \Delta \psi_t(x) + \psi_t(x)\, \eta_t(x) \tag{1.1}$$

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where $\psi_t = \psi_t(x)$, $t \geqslant 0$, is a scalar field in $\mathbf{R}^1$, Δ is the Laplacian, ν is a positive constant, and $\eta_t = \eta_t(x)$ is a two-parameter white noise, i.e.,

$$\mathbf{E}(\eta_t(x)\,\eta_{t'}(x')) = \delta(t-t')\,\delta(x-x') \tag{1.2}$$

Equation (1.1), often called the stochastic heat equation, is a linearized model for the evolution of a scalar field in a space-time-dependent random medium.[13] The parameter ν then has the interpretation of the viscosity coefficient. The choice of the white noise as random potential corresponds to considering those regimes with very rapid variations, the type of turbulent flows.

A traditional way to investigate the evolution of the field ψ_t is to study its moments. This method is important not only because it is constructive, but also because the moments themselves have a physical meaning, which is often more important than that of the individual solution. Molchanov,[13] studying the moments of the solution of Eq. (1.1) on a lattice, $(t, x) \in \mathbf{R}^+ \times \mathbf{Z}^d$, shows that the field ψ_t has an intermittent behavior. From a qualitative point of view intermittent random fields are characterized by the appearance of sharp peaks which give the main contribution to the statistical moments.

Our analysis extends the general picture in ref. 13 to Eq. (1.1) and improves some quantitative results. In the continuum case, the random potential is singular and a rigorous analysis of (1.1) is not completely trivial. In particular, white noise gives the same weight to all scales, without introducing any characteristic length or time. The physical requirement behind this choice is that the solution of (1.1), which is supposed to describe macroscopic phenomena, should not be too sensitive with respect to fluctuations occurring at arbitrary small scales. Due to the singularity of white noise, our results are, however, restricted to one space dimension.

In this paper we construct the solution of the Cauchy problem associated with Eq. (1.1) via a generalized Feynman–Kac formula. The initial data are in a set in which distributions also are enclosed. In particular we are interested in initial functions which are either localized (δ-type initial conditions) or spatially homogeneous (constant initial conditions). For the former case the stochastic evolution has the effect of smoothing the singularity: we prove that for any $t > 0$ the process $\psi_t(x)$ is continuous in the space variable regardless of the initial data.

The Feynman–Kac expression allows us a rather complete analysis of the statistical properties of the solution. The two-point correlation function is explicitly computed. We then study asymptotic (in time) properties of the solution. In particular we focus on translation-invariant initial data, i.e., we assume $\psi_0(x) = \text{const}$, and evaluate the moments of the process $\psi_t(x)$. This result establishes that the solution of Eq. (1.1), following the definition given by Molchanov,[13] has an intermittent behavior. They key point is a representation of the statistical moments in terms of local times for independent Brownian motions. This representation permits us to carry out the computations and obtain explicit formulas.

Equation (1.1) arises in several other physical problems: it is satisfied by the partition function of a directed polymer in a two-dimensional random medium described by the random potential η.[11] It is furthermore related to the random growth of interfaces and to the Burgers equation with noise: introducing (Cole–Hopf transformation) $h_t(x) := \nu \log \psi_t(x)$, it satisfies

$$\partial_t h_t(x) = \frac{\nu}{2} \Delta h_t(x) + \frac{1}{2} (\partial_x h_t(x))^2 + \nu \eta_t(x) \tag{1.3}$$

which is the so-called KPZ equation[10] proposed as a (nonlinear) random model of growing interfaces. Here $h_t(x)$ is the height of the interface and ν the surface tension. By differentiating (1.3) and defining $u_t(x) := -\partial_x h_t(x)$, we get the Burgers equation with conservative noise

$$\partial_t u_t(x) = \frac{\nu}{2} \Delta u_t(x) - \partial_x \left[\frac{1}{2} u_t(x)^2 + \nu \eta_t(x) \right] \tag{1.4}$$

which has been largely studied in the physical literature as a simplified model in complex phenomena such as turbulence, intermittence, and large-scale structure. A satisfactory mathematical theory of Eqs. (1.3) and (1.4) is, however, lacking. See Remark 3 after Theorem 2.2 for a further discussion. The relationship of (1.4) to (1.1) is also exploited in ref. 8, where white noise analysis techniques are used. The less singular problem of the Burgers equation with nonconservative space-time white noise is studied in refs. 2 and 6.

To study rigorously the stochastic heat equation, we realize the white noise as the (generalized) derivative of a Wiener process: $\eta_t = \partial_t B_t$. We can thus rewrite Eq. (1.1) as

$$d\psi_t = \frac{\nu}{2} \Delta \psi_t \, dt + \psi_t \, dB_t \tag{1.5}$$

Since it contains a nontrivial diffusion, the stochastic differential $\psi_t \, dB_t$ presents the well-known ambiguities. The correct choice is not a trivial point. In ref. 2, for example, a similar equation, where the random potential is the space integral of white noise, has been studied. It is there shown how, in order to obtain that the Cole–Hopf transformation of ψ_t gives a solution of the Burgers equation, the stochastic differential has to be interpreted in the Stratonovich sense. In the present case, as the random potential is more singular, the situation is more complicated. The Feynman–Kac formula for the linear equation (1.5) when the stochastic differential is interpreted in the Stratonovich sense is not well defined. However, after a simple renormalization—the Wick exponential—a meaningful expression is obtained. This renormalized Feynman–Kac formula solves Eq. (1.5) when the stochastic differential is interpreted in the Ito sense. When the Cole–Hopf transformation is performed this implies a Wick renormalization of the nonlinear term in Eqs. (1.3) and (1.4).[3,5]

We note that Eq. (1.5) in any dimension, with a noise regular in the space variable, has been studied in ref. 16, where the stochastic differential is interpreted in the Stratonovich sense and, more recently, in ref. 15 with both interpretations of the stochastic differential. The latter paper also discusses the white noise case in one space dimension.

The paper is structured as follows. In the next section we introduce the mathematical apparatus and state precisely our results. In particular we review in some detail what is meant by intermittence and recall the basic definitions and properties of local times.

In Section 3 we prove the Feynman–Kac formula; this allows us to establish an existence and uniqueness theorem for the Cauchy problem for the stochastic heat equation. We also prove some smoothness results for the realizations of the process. The representation in terms of local times is introduced and used for a technical bound.

Section 4 is devoted to the proof of the statistical properties; here the representation in terms of local times plays a more fundamental role: using known results on their distribution, we reduce the proofs to straightforward computations.

2. PRELIMINARIES AND RESULTS

2.1. Wiener Process and Stochastic Integrals

Let B_t, $t \geqslant 0$, be the cylindrical Wiener process on $L^2(\mathbf{R}, dx)$. It is realized as a distribution-valued continuous process, i.e., the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ is given by $\Omega = C(\mathbf{R}^+; \mathscr{S}')$; here $\mathscr{S}'$ is the Schwartz space of distribution on $\mathbf{R}$, $\mathscr{F}$ is the σ-algebra generated by the cylindrical sets, and $\mathscr{P}$ is the Gaussian measure with covariance

$$\mathbf{E}(B_t(f)\, B_{t'}(g)) = t \wedge t'(f, g) \tag{2.1}$$

where $f, g \in \mathscr{S}$ are test functions, $a \wedge b = \min\{a, b\}$, and $(\cdot, \cdot)$ is the scalar product in $L^2(\mathbf{R}, dx)$. We denote by $\mathscr{F}_t$ the natural filtration of B_t, i.e., the minimal σ-algebra such that $s \mapsto B_s$ is $\mathscr{F}_t$ measurable for all $s \in [0, t]$.

Let λ_t, $t > 0$, an $L^2(\mathbf{R}, dx)$-valued, $\mathscr{F}_t$-adapted continuous process such that for any $t > 0$

$$\mathbf{E} \int_0^t ds\, (\lambda_s, \lambda_s) < \infty \tag{2.2}$$

We can then define the Ito integral of λ_t with respect to the Wiener process as

$$\int_0^t (\lambda_s, dB_s) := \sum_{i=1}^{\infty} \int_0^t (\lambda_s, e_i)\, dB_s(e_i) \tag{2.3}$$

where $\{e_i\}$ is an orthonormal basis in $L^2(\mathbf{R}, dx)$ and thus $\{B_t(e_i)\}$ are independent one-dimensional Wiener processes; the series is convergent, in $L^2(\mathscr{P})$, by (2.2).

We need a regularized version of B_t, which is defined as follows. Let $h \in C_0^\infty(\mathbf{R})$ be an even positive function such that $\int dx\, h(x) = 1$. Introduce, for $\kappa > 0$, the mollifier $\delta_\kappa(x) := \kappa h(\kappa x)$ and define $B_t^\kappa(x) := B_t(\delta_k(x - \cdot))$. Its correlation function is

$$\mathbf{E}(B_t^\kappa(x)\, B_{t'}^{\kappa'}(x')) = t \wedge t' C_{\kappa,\kappa'}(x - x'), \qquad C_{\kappa,\kappa'} := \delta_\kappa \star \delta_{\kappa'} \tag{2.4}$$

where $\star$ denotes convolution in space; if $\kappa = \kappa'$, we use the notation $C_\kappa := C_{\kappa,\kappa}$.

For κ finite, B_t^κ is a nice [e.g., $C^\infty(\mathbf{R})$ valued] process; our results will be obtained letting $\kappa \to \infty$ and showing we have meaningful expressions also in the limit.

To construct the solution of equation (1.5) through a Feynman–Kac formula, we need a stochastic curvilinear integral. We now define it for the regularized process B_t^κ. Let $s \mapsto \varphi_s$ be a Hölder continuous function from $[0, \infty)$ to $\mathbf{R}$ and $s_i = 2^{-n} it$, $i = 0,..., 2^n$, be a partition of $[0, t]$; introduce

$$M_\varphi^{\kappa,n}(t) := \sum_i \left(B_{s_{i+1}}^\kappa(\varphi_{s_i}) - B_{s_i}^\kappa(\varphi_{s_i})\right) \tag{2.5}$$

It is not difficult to verify that $M_\varphi^{\kappa,n}(t)$ is a Cauchy sequence in $L^2(\mathscr{P})$; its limit defines

$$M_\varphi^\kappa(t) := \lim_{n \to \infty} M_\varphi^{\kappa,n}(t), \qquad t \in [0, \infty) \tag{2.6}$$

which is, under $\mathscr{P}$, a continuous Gaussian process and an $\mathscr{F}_t$ Martingale.

If $s \mapsto \gamma_s$ is another function, the cross-variation of $M_\varphi^\kappa(t)$ and $M_\gamma^{\kappa'}(t)$ is

$$\langle M_\varphi^\kappa, M_\gamma^{\kappa'} \rangle_t = \int_0^t ds\, C_{\kappa,\kappa'}(\varphi_s - \gamma_s) \tag{2.7}$$

We note that, since $B_t^\kappa(x)$ is Lipschitz in x, this construction is a particular case of the general theory developed in ref. 7.

Remark. In ref. 2 an analogous stochastic curvilinear integral was defined for the Brownian sheet; in that case it was proven to be meaningful for the nonregularized process. As can be seen from (2.7) the variance of $M_\varphi^\kappa(t)$ is now $tC_\kappa(0)$, which diverges when $\kappa \to \infty$. As we shall see, this is the reason why the Feynman–Kac formula for the linear equation (1.5) with the Stratonovich stochastic differential needs a renormalization.

2.2. Formulation of the Cauchy Problem and Feynman–Kac Formula

Let us introduce the heat kernel

$$G_t(x) := \frac{1}{(2\nu\pi t)^{1/2}} \exp\left(-\frac{x^2}{2\nu t}\right) \tag{2.8}$$

We assume the initial datum ψ_0 to be a positive Borel measure on $\mathbf{R}$ such that, defining

$$G_t \star \psi_0(x) := \int d\psi_0(y)\, G_t(x-y) \tag{2.9}$$

it satisfies

$$\sup_{t \in (0,T]} \sup_{x \in \mathbf{R}} \sqrt{t}\, G_t \star \psi_0(x) < \infty \tag{2.10}$$

for any $T > 0$. This allows a singularity of order $t^{-1/2}$ as $t \to 0^+$ and permits a delta-type initial condition. However, the application $(t, x) \mapsto G_t \star \psi_0(x)$ is smooth, e.g., differentiable, for any $t > 0$, $x \in \mathbf{R}$.

In the study of the statistical properties we will focus on the cases of the Lebesgue and Dirac measures.

We now formulate the Cauchy problem for the stochastic heat equation (1.5), as an Ito equation, in a convenient mild form.

Definition 2.1. Let $\psi_t = \psi_t(x)$, $t > 0$, be a continuous, $\mathscr{F}_t$-adapted process such that for any $T > 0$

$$\sup_{t \in (0, T]} \sup_{x \in \mathbf{R}} \int_0^t ds \int_0^s ds' \int dy\, dy'\, G_{t-s}(x-y)^2 \times G_{s-s'}(y-y')^2\, \mathbf{E}(\psi_{s'}(y')^2) < \infty \tag{2.11}$$

it is a *solution* of the stochastic heat equation if for any $t > 0$

$$\psi_t = G_t \star \psi_0 + \int_0^t G_{t-s} \star \psi_s\, dB_s \qquad \mathscr{P}\text{-a.s.} \tag{2.12}$$

where

$$\int_0^t G_{t-s} \star \psi_s\, dB_s(x) := \int_0^t (G_{t-s}(x - \cdot)\psi_s, dB_s) \tag{2.13}$$

is the Ito integral defined in (2.3).

We remark that, even if the initial datum ψ_0 is a measure, we have formulated the stochastic heat equation for processes which are, for any $t > 0$, function-valued and satisfy (2.11). We will actually prove that the solution is $C^0(\mathbf{R})$-valued as $t > 0$.

The initial datum ψ_0 is satisfied in the distribution sense. In fact, using (2.11) and (2.12), it can be verified that if ψ_t is a solution of the stochastic heat equation, for any $f \in C^0(\mathbf{R})$ and uniformly bounded,

$$\lim_{t \to 0^+} \int dx\, f(x)\, \psi_t(x) = \int d\psi_0(x)\, f(x) \tag{2.14}$$

where the limit is $\mathscr{P}$-a.s.

We first define precisely the Feynman–Kac formula at the level of the regularized Wiener process B_t^κ. Let b_s, $s \in [0, t]$, be the Brownian bridge, with diffusion coefficient ν, between y and x; i.e., the Gaussian process with mean $y + (x-y)st^{-1}$ and covariance $\Gamma(s', s) = \nu t^{-1} s'(t-s)$ where $s' \leqslant s$. In particular, $b_0 = y$, $b_t = x$. We denote by $P^b_{y,x;t}$ the law of b; we write $P^{b,\nu}_{y,x;t}$ when we want to indicate explicitly the dependence on ν. We stress that b is independent of the cylindrical Wiener process B.

Let finally

$$dP^b_{x,t} := \int d\psi_0(y)\, G_t(x-y)\, dP^b_{y,x;t} \tag{2.15}$$

The expectations with respect to $dP^b_{x,y;t}$ and $dP^b_{x,t}$ are denoted by $\mathbf{E}^b_{x,y;t}$ and $\mathbf{E}^b_{x,t}$, respectively. They are not to be confused with the expectation with respect to $\mathscr{P}$, denoted by $\mathbf{E}$.

Let us consider the regularized form of Eq. (2.12),

$$\psi^\kappa_t = G_t \star \psi_0 + \int_0^t G_{t-s} \star \psi^\kappa_s \, dB^\kappa_s \tag{2.16}$$

Its solution can be expressed, as shown in the next section, by the following generalized Feynman–Kac formula:

$$\psi^\kappa_t(x) := \mathbf{E}^b_{x,t}\, \mathscr{E}\mathrm{xp}\{M^\kappa_b(t)\} \tag{2.17}$$

where $M^\kappa_b(t)$ is defined pathwise $dP^b_{x,t}$-a.s. by (2.6) and

$$\mathscr{E}\mathrm{xp}\{M^\kappa_b(t)\} := \exp\{M^\kappa_b(t) - \tfrac{1}{2}\langle M^\kappa_b, M^\kappa_b\rangle_t\} = \exp\{M^\kappa_b(t) - \tfrac{1}{2} t C_\kappa(0)\} \tag{2.18}$$

in the martingale terminology is the Girsanov exponential of M^κ_b, or the Wick exponential in the language of quantum field theory. In our context both of these characterizations are useful. The diverging term $C_\kappa(0)$ provides the aforementioned renormalization on the Feynman–Kac formula and a meaningful expression is obtained in the limit $\kappa \to \infty$.

Theorem 2.2. For any $t > 0$, $x \in \mathbf{R}$, $\psi^\kappa_t(x)$ defined in (2.17) is a Cauchy sequence in $L^2(\mathscr{P})$; denoting by $\psi_t = \psi_t(x)$ its limit, we have:

(i) For all $p \geq 1$, $\psi^\kappa_t(x) \to \psi_t(x)$ in $L^p(\mathscr{P})$ and $\mathscr{P}$-a.s. The convergence is uniform for $x \in \mathbf{R}$ and for t in compact subsets of $(0, \infty)$.

(ii) ψ_t is the unique solution of the stochastic heat equation as formulated in Definition 2.1.

(iii) For $(t, x) \in (0, \infty) \times \mathbf{R}$, $(t, x) \mapsto \psi_t(x)$ is $\mathscr{P}$-a.s. Hölder continuous. The Hölder exponent is $\alpha < 1/2$ in space and $\alpha < 1/4$ in time.

(iv) For any $(t, x) \in (0, \infty) \times \mathbf{R}$, $\psi_t(x) > 0$ $\mathscr{P}$-a.s.

The key point in the proof of the theorem is to establish that $\psi^\kappa_t(x)$ is a Cauchy sequence. The important statement (iv) is essentially contained in Mueller,[14] to which we will refer.

Remark 1. We have considered, for notational simplicity, deterministic initial data; however, our results are easily extended to random initial data.

Remark 2. We have considered positive initial data because in the physical problems we have outlined one is mostly interested in positive solutions of (2.12); however, as the equation is linear, the solution with signed initial datum can be constructed by superposition.

Remark 3. By the results in Theorem 2.2, we can construct, as a $C^0(\mathbf{R})$-valued process, $h_t(x) := v \log \psi_t(x)$, which describes the interface growth in the KPZ model (1.3). Analogously the random field for the Burgers equation with conservative noise can be rigorously defined, as a distribution-valued process, by

$$u_t(f) := v \int dx\, f'(x) \log \psi_t(x) \tag{2.19}$$

where f is a test function. However, as the nonlinear terms in Eqs. (1.3) and (1.4) involve ill-defined operations between distributions, those equations do not have a rigorous meaning when the cutoff is removed.

2.3. Local Times and Statistical Properties

We recall the basic definitions and properties of the local times; for a comprehensive discussion see, e.g., ref. 17, a book that will be quoted when we need specific results. Given a continuous semi-martingale X and $a \in \mathbf{R}$ there exists an increasing process $L_t^a(X)$, called the local time of X in a, such that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\, dX_s + L_t^a(X) \tag{2.20}$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x \leqslant 0$.

The process $L_t^a(X)$ can be described informally as $\int_0^t \delta(X_s - a)\, d\langle X, X\rangle_s$, where $\delta(\cdot)$ is the Dirac delta function. The following result is instead rigorously proven:

$$L_t^a(X) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{(a, a+\varepsilon)}(X_s)\, d\langle X, X\rangle_s \tag{2.21}$$

where $\mathbf{1}_A$ is the characteristic function of the set A and the limit is almost surely.

We will consider only the local times of the Brownian bridge and of the Brownian motion; in both cases $d\langle X, X\rangle_t = v\, dt$. For notational convenience we define the local times using the measure dt in (2.21), so that our local times are v^{-1} the usual ones; thus the local time $L_t^a(X)$ measures (with respect to Lebesgue) the time that X has spent in a. We shall use the notation $L_t(X) := L_t^0(X)$.

The statistical moments of the field ψ_t can be expressed through local times of Brownian bridges as stated in the following proposition.

Proposition 2.3. Let $\vec{b}_s = (b_s^1,..., b_s^n)$, $s \in [0, t]$, be n independent Brownian bridges between $\vec{y} = (y_1,..., y_n)$ and x. Then

$$\mathbf{E}(\psi_t(x)^n) = \int \prod_{i=1}^{n} d\psi_0(y_i)\, G_t(x - y_i) \cdot \mathbf{E}^{\vec{b}, \nu}_{\vec{y}, x; t}\left(\exp \sum_{i<j} L_t(b^i - b^j)\right) \tag{2.22}$$

The two-point correlation function is given, for $t \leqslant t'$, by

$$\mathbf{E}(\psi_t(x)\, \psi_{t'}(x')) = \int d\psi_0(y)\, d\psi_0(y')\, dz\, G_t(x-y)\, G_{t'}(x'-y')$$
$$\times G_{t'-t}(z)\, \mathbf{E}^{b, 2\nu}_{y'-y, x'-x+z; t}(e^{L_t(b)}) \tag{2.23}$$

Let us introduce the notation

$$\Phi(\xi) := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\xi} dy\, e^{-y^2/2} \tag{2.24}$$

for the Gaussian distribution.

When the initial datum is either the Lebesgue or the Dirac measure, Proposition 2.3 has the following corollaries.

Corollary 2.4. If ψ_0 is the Lebesgue measure, we have

$$\mathbf{E}(\psi_t(x)\, \psi_t(x')) = \int_0^t ds \frac{|x-x'|}{(\pi \nu s^3)^{1/2}} \exp\left[-\frac{(x-x')^2}{4\nu s} + \frac{t-s}{4\nu}\right] \Phi\left(\left(\frac{t-s}{2\nu}\right)^{1/2}\right) \tag{2.25}$$

Corollary 2.5. If ψ_0 is the Dirac measure in 0, we have

$$\mathbf{E}(\psi_t(x)\, \psi_t(x')) = \frac{1}{2\pi\nu t} \exp\left[-\frac{x^2 + (x')^2}{2\nu t}\right] \int_0^1 ds \frac{|x-x'|}{(4\pi\nu t)^{1/2}} \frac{1}{[s^3(1-s)]^{1/2}}$$
$$\times \left\{\exp\left[-\frac{(x-x')^2}{4\nu t}\frac{1-s}{s}\right]\right\}\left\{1 + \left[\frac{\pi t(1-s)}{\nu}\right]^{1/2}\right.$$
$$\left.\times \left[\exp\left(\frac{t}{2\nu}\frac{1-s}{2}\right)\right] \Phi\left(\left(\frac{t(1-s)}{2\nu}\right)^{1/2}\right)\right\} \tag{2.26}$$

In particular,

$$\mathbf{E}(\psi_t(x)^2) = \frac{1}{2\pi\nu t} e^{-x^2/\nu t} \left[1 + \left(\frac{\pi t}{\nu}\right)^{1/2} e^{t/4\nu} \Phi\left(\left(\frac{t}{2\nu}\right)^{1/2}\right)\right] \tag{2.27}$$

2.4. Moment Lyapunov Exponents and Intermittence

Before stating our result, we briefly review the moment approach to intermittence in space-time-dependent random media.[1,4,13]

Let the process $\varphi_t(x)$ be homogeneous and ergodic with respect to translation of the space variable; its moments

$$m_n(t) := \mathbf{E}(\varphi_t(x)^n) \tag{2.28}$$

do not depend on x. The n-moment Lyapunov exponent can be defined if the limit

$$\gamma_n := \lim_{t \to \infty} \frac{\log m_n(t)}{t} \tag{2.29}$$

is finite. The process φ_t is *intermittent* if the strict inequalities

$$\gamma_1 < \frac{1}{2}\gamma_2 < \cdots < \frac{1}{n}\gamma_n < \cdots \tag{2.30}$$

are satisfied.

To explain the rationale behind this definition, let $\alpha \in (\gamma_1, \gamma_2/2)$ and consider the following random set:

$$B_{t,\alpha} := \{x: \varphi_t(x) > e^{\alpha t}\} \tag{2.31}$$

The ergodic theorem ensures that the volume density of this set

$$\rho_{t,\alpha} := \lim_{R \to \infty} \frac{\mathrm{Vol}(B_{t,\alpha} \cap \{|x| < R\})}{\mathrm{Vol}(\{|x| < R\})} \tag{2.32}$$

exists and is given by $P\{\varphi_t(x) > e^{\alpha t}\}$. By the Chebyshev inequality we then have

$$\rho_{t,\alpha} = P\{\varphi_t(x) > e^{\alpha t}\} \leqslant e^{-\alpha t}\mathbf{E}(\varphi_t(x)) \stackrel{\cdot}{\sim} e^{-(\alpha - \gamma_1)t} \tag{2.33}$$

The notation $\stackrel{\cdot}{\sim}$ denotes logarithmic equivalence, i.e.,

$$f(t) \stackrel{\cdot}{\sim} g(t) \Leftrightarrow \lim_{t \to \infty} \frac{\log f(t) - \log g(t)}{t} = 0 \tag{2.34}$$

For large t the density of the set $B_{t,\alpha}$ is thus exponentially small.

The second moment can be written as

$$m_2(t) = \mathbf{E}(\varphi_t^2(x)) = \mathbf{E}(\varphi_t^2(x)\mathbf{1}_{B_{t,\alpha}}) + \mathbf{E}(\varphi_t^2(x)\mathbf{1}_{\mathbf{R}\setminus B_{t,\alpha}}) \tag{2.35}$$

where $\mathbf{1}_{B_{t,\alpha}}$ is the indicator of the set $B_{t,\alpha}$. The second term in (2.35) does not exceed $e^{2\alpha t}$; furthermore, $e^{2t\alpha} \ll e^{\gamma_2 t}$; hence

$$m_2(t) \sim \mathbf{E}(\varphi_t^2(x)\mathbf{1}_{B_{t,\alpha}}) \tag{2.36}$$

Therefore the second moment is generated almost entirely by the sharp fluctuations of the field $\varphi_t(x)$ concentrated in the set $B_{t,\alpha}$, whose density, as we saw above, is exponentially small.

In the same way, choosing a parameter sequence $\{\alpha_n\}$ such that

$$\frac{1}{n}\gamma_n < \alpha_n < \frac{1}{n+1}\gamma_{n+1} \tag{2.37}$$

we obtain a hierarchical sequence of sets

$$B_{t,\alpha_1} \supset B_{t,\alpha_2} \supset B_{t,\alpha_3} \supset \cdots \tag{2.38}$$

Each of them is a collection of small islands, the distribution of which is exponentially small. Repeating the same argument for the second moment, it is easy to understand how every moment is generated by the values that the process $\varphi_t(x)$ assumes in the corresponding set of the hierarchy. This shows how the strict inequalities (2.30) imply the presence of a peculiar local structure, hence the name *intermittence.*

We now discuss the moment Lyapunov exponents for the stochastic heat equation. We consider deterministic translation-invariant initial data, i.e., we assume ψ_0 to be the Lebesgue measure. For such initial datum we can state the following theorem.

Theorem 2.6. The nth moment of $\psi_t(x)$ is given by

$$\mathbf{E}(\psi_t(x)^n) = 2\exp\left\{\frac{n(n^2-1)}{4!\,\nu}t\right\}\Phi\left(\left(\frac{n(n^2-1)}{12\nu}t\right)^{1/2}\right) \tag{2.39}$$

In particular, the nth-moment Lyapunov is

$$\gamma_n = \frac{1}{4!\,\nu}n(n^2-1) \tag{2.40}$$

Remark. In the directed polymer case one is interested in a delta initial condition $\psi_0 = \delta_0$ and in evaluating the moments of $\bar{\psi}_t := \int dx\, \psi_t(x)$; see ref. 12. They are still given by formula (2.39).

The cylindrical Wiener process $B_t(x)$ is homogeneous and ergodic with respect to translation of the space variable and we are considering homogeneous and ergodic initial datum with respect to spatial translations, so the process $\psi_t(x)$, being a functional of $B_t(x)$, is homogeneous and ergodic. We can thus conclude, by Theorem 2.6, that for each (positive) value of the parameter ν, the process $\psi_t(x)$ has an intermittent behavior.

As we remarked in the introduction, the stochastic heat equation has been extensively studied on a lattice.[13] The discrete case has some differences from the continuous one. The renormalization term $C_\kappa(0)$ is finite and no regularization is needed in constructing the solution. All the Lyapunov exponents ($n > 2$) are estimated as functions of γ_2, of which there is not an explicit expression, but only the qualitative behavior as a function of the viscosity ν: it tends to one in the limit $\nu \to 0$ and to zero in the limit $\nu \to \infty$. In the continuum case instead, the moment Lyapunov exponents diverge in the limit $\nu \to 0$.

We note formula (2.40) has been obtained in ref. 9, showing that the nth-moment Lyapunov exponent is given by the lowest eigenvalue of an n-body Schrödinger operator with a two-body delta potential; the result (2.40) is obtained when the self-interactions are ignored. Our approach is instead purely probabilistic: the use of local times permits an exact and rigorous calculation of the statistical moments, from which the Lyapunov exponents are then obtained as leading order. Furthermore, the discussion of the stochastic differential and the renormalization of the Feynman–Kac formula give a clear mathematical meaning to the physical hypothesis of ignoring self-interactions.

In the physical literature[9, 12] the free energy of a directed polymer in a random environment is obtained from (2.40) via the replica method. The exact formula (2.39) shows explicitly the problems connected with the analytic continuation: the argument of Φ becomes imaginary if $n \in (0, 1)$.

3. FEYNMAN–KAC FORMULA

This section is devoted to the proof of Theorem 2.2. We first show that the Feynman–Kac formula (2.17) is meaningful when $\kappa \to \infty$. We next prove that the limiting process is the unique solution of the stochastic heat equation. Finally we establish the Hölder continuity of the trajectories. The representation in terms of local times is introduced and used to obtain the necessary bounds on the moments of ψ_t^κ.

The first step in the proof of Theorem 2.2 consists in verifying that $\psi_t^\kappa(x)$ defined in (2.17) is a Cauchy sequence in $L^2(\mathscr{P})$. Let $dP^{b'}_{x,t}$ be an independent copy of the measure $dP^b_{x,t}$. We have

$$\mathbf{E}(\psi_t^\kappa(x)-\psi_t^{\kappa'}(x))^2=\mathbf{E}^b_{x,t}\mathbf{E}^{b'}_{x,t}\mathbf{E}[(\mathscr{E}\mathrm{xp}\{M_b^\kappa(t)\}-\mathscr{E}\mathrm{xp}\{M_b^{\kappa'}(t)\}) \times(\mathscr{E}\mathrm{xp}\{M_{b'}^\kappa(t)\}-\mathscr{E}\mathrm{xp}\{M_{b'}^{\kappa'}(t)\})] \tag{3.1}$$

Since $M_b^\kappa(t)$, under $\mathscr{P}$, is a Gaussian variable, the expectation can be explicitly computed; recalling (2.7), we get

$$\begin{aligned}
&\mathbf{E}^B(\psi_t^\kappa(x)-\psi_t^{\kappa'}(x))^2\\
&\quad=\mathbf{E}^b_{x,t}\mathbf{E}^{b'}_{x,t}\left[\exp\int_0^t ds\, C_\kappa(b_s-b'_s)-2\exp\int_0^t ds\, C_{\kappa,\kappa'}(b_s-b'_s)\right.\\
&\qquad\left.+\exp\int_0^t ds\, C_{\kappa'}(b_s-b'_s)\right]\\
&\quad=\int d\psi_0(y)\, d\psi_0(y')\, G_t(x-y)\, G_t(x-y')\\
&\qquad\times\mathbf{E}^{b,2\nu}_{y-y',0;t}\left[\exp\int_0^t ds\, C_\kappa(b_s)-2\exp\int_0^t ds\, C_{\kappa,\kappa'}(b_s)\right.\\
&\qquad\left.+\exp\int_0^t ds\, C_{\kappa'}(b_s)\right]
\end{aligned} \tag{3.2}$$

as $b_s-b'_s$ is, in law, the Brownian bridge from $y-y'$ to 0 with diffusion coefficient 2ν.

Let

$$L_t^\kappa(b):=\int_0^t ds\, C_\kappa(b_s) \tag{3.3}$$

The proof that the right-hand side of (3.2) converges to 0 as $\kappa,\kappa'\to\infty$ will be completed after the next two lemmata, which provide the necessary bounds on $L_t^\kappa(b)$. They are based on elementary properties of the local times.

Lemma 3.1. Let $L_t(b)$ be as defined in (2.21). For any $p\in[1,\infty)$, there exists a constant $c>0$ such that for all $t\in[0,\infty)$

$$\sup_{z\in\mathbf{R}}\|L_t^\kappa(b)-L_t(b)\|_{L^p(dP^b_{z,0;t})}\leqslant ct^{1/4}k^{-1/2} \tag{3.4}$$

Proof. By scaling we have, in law, $L_t(b) = t^{1/2} L_1(\tilde{b})$, where $\tilde{b}_s$, $s \in [0, 1]$, is the Brownian bridge between $zt^{-1/2}$ and 0. On the other hand, introducing $\tilde{h} := h \star h$ and recalling the definition of C_κ, we have

$$L_t^\kappa(b) = \kappa \int_0^t ds\, \tilde{h}(\kappa b_s) \overset{\text{Law}}{=} t\kappa \int_0^1 ds\, \tilde{h}(t^{1/2} \kappa \tilde{b}_s) \tag{3.5}$$

Let us recall the occupation time formula (ref. 17, Chapter VI, 1.6), which states that for any positive Borel-measurable function f, the following identity holds $dP^b_{x,0;t}$-a.s.:

$$\int_0^t ds\, f(b_s) = \int da\, L_t^a(b)\, f(a) \tag{3.6}$$

Using the identity (3.5) and the occupation time formula (3.6) and recalling the normalization $\int da\, \tilde{h}(a) = 1$, we have

$$\begin{aligned}
&\| L_t^\kappa(b) - L_t(b) \|_{L^p(dP^b_{z,0;t})} \\
&\quad = t^{1/2} \left\| t^{1/2}\kappa \int da\, \tilde{h}(t^{1/2}\kappa a)\, L_1^a(b) - L_1(b) \right\|_{L^p(dP^b_{zt^{-1/2},0;1})} \\
&\quad \leqslant t^{1/2} \int da\, \tilde{h}(a)\, \| L_1^{a(t^{1/2}\kappa)^{-1}}(b) - L_1(b) \|_{L^p(dP^b_{zt^{-1/2},0;1})} \\
&\quad \leqslant c_1 t^{1/2} \int da\, \tilde{h}(a) (a(t^{1/2}\kappa)^{-1})^{1/2} = c t^{1/4} \kappa^{-1/2}
\end{aligned} \tag{3.7}$$

where, in the last inequality, we used the L^p Hölder continuity of exponent 1/2 of the local time of the Brownian bridge (ref. 17, Chapter VI, 1.8). ∎

Lemma 3.2. For any $p > 0$, $T > 0$ there exists a constant $c > 0$ such that, for any $\kappa > 0$,

$$\sup_{z \in \mathbf{R},\, t \in [0, T]} \mathbf{E}^b_{z,0;t}(e^{pL_t^\kappa(b)}) \leqslant c \tag{3.8}$$

Proof. Retaining the notations introduced in the previous lemma, by scaling and the occupation time formula (3.6), we have

$$\begin{aligned}
\mathbf{E}^b_{z,0;t}(\exp[pL_t^\kappa(b)]) &= \mathbf{E}^b_{zt^{-1/2},0;1}\left(\exp\left[pt^{1/2} \int da\, \tilde{h}(a)\, L_1^{a(t^{1/2}\kappa)^{-1}}(b)\right]\right) \\
&\leqslant \mathbf{E}^b_{zt^{-1/2},0;1} \int da\, \tilde{h}(a) \exp[pt^{1/2} L_1^{a(t^{1/2}\kappa)^{-1}}(b)]
\end{aligned} \tag{3.9}$$

where we used the Jensen inequality.

The bound (3.8) is then proven once we show that there exists $c_1 > 0$ such that for each $\lambda \in [0, \Lambda]$

$$\sup_{a, z \in \mathbf{R}} \mathbf{E}^b_{z,0;1}(\exp[\lambda L^a_1(b)]) \leqslant c_1 \tag{3.10}$$

The intuition about the local times suggests that the supremum in (3.10) is attained for $a = z = 0$; a computation shows that then the expectation is finite. We will prove this fact, introducing appropriate stopping times and reducing to the local time in 0 for the Brownian bridge from 0 to 0.

Let us introduce, for the Brownian bridge b starting from z and arriving in 0 at time 1, the stopping time $T_a := \{\inf t: b_t = a\}$ and denote by $P_{a,z}$ its distribution. Using the strong Markov property and the additivity of the local time, we have

$$\begin{aligned} \mathbf{E}^b_{z,0;1}(e^{\lambda L^a_1(b)}) &= \int_0^1 P_{a,z}(dt)\, \mathbf{E}^b_{a,0;1-t}(e^{\lambda L^a_{1-t}(b)}) \\ &= \int_0^1 P_{a,z}(dt)\, \mathbf{E}^{\tilde{b}}_{0,a;1-t}(e^{\lambda L^a_{1-t}(\tilde{b})}) \end{aligned} \tag{3.11}$$

In the last identity we used the time-reversal property of the Brownian bridge, i.e., if b_s, $s \in [0, \tau]$, is a Brownian bridge from a to 0, then $\tilde{b}_s := b_{\tau - s}$, is, in law, a Brownian bridge from 0 to a.

We now introduce, for the Brownian bridge $\tilde{b}$ starting from 0 and arriving in a at time $1-t$, the stopping time $\tilde{T}_a := \{\inf s: \tilde{b}_s = a\}$ and denote by $\tilde{P}_{a,t}$ its distribution.

We can then write (3.11) as

$$\begin{aligned} \mathbf{E}^b_{z,0;1}(e^{\lambda L^a_1(b)}) &= \int_0^1 P_{a,z}(dt) \int_0^{1-t} \tilde{P}_{a,t}(ds)\, \mathbf{E}^{\tilde{b}}_{a,a;1-t-s}(e^{\lambda L^a_{1-t-s}(\tilde{b})}) \\ &= \int_0^1 P_{a,z}(dt) \int_0^{1-t} \tilde{P}_{a,t}(ds)\, \mathbf{E}^b_{0,0;1}(e^{\lambda (1-t-s)^{1/2} L_1(b)}) \end{aligned} \tag{3.12}$$

The last identity is obtained by translation and scaling.

The right-hand side of (3.12) can now be bounded using the following result (see ref. 17, Chapter XII, 3.8). If b_s, $s \in [0, 1]$, is a Brownian bridge (with diffusion coefficient ν) from 0 to 0, then, in law, $L_1(b) = (2\gamma)^{1/2}$, where γ is an exponential random variable with mean ν^{-1}. ∎

Proof of Theorem 2.2. We conclude the proof that $\psi^\kappa_t(x)$ in (2.17) is a Cauchy sequence in $L^2(\mathscr{P})$. By Lemmata 3.1 and 3.2, we have

$$\mathbf{E}^{b,2\nu}_{y-y',0;t}\left|\exp\int_0^t ds\, C_\kappa(b_s)-\exp\int_0^t ds\, C_{\kappa,\kappa'}(b_s)\right|$$
$$\leqslant\left\|\exp\int_0^t ds\, C_\kappa(b_s)+\exp\int_0^t ds\, C_{\kappa,\kappa'}(b_s)\right\|_{L^2(dP^{b,2\nu}_{y-y';0;t})}$$
$$\times\left\|\int_0^t ds\, C_\kappa(b_s)-\int_0^t ds\, C_{\kappa,\kappa'}(b_s)\right\|_{L^2(dP^{b,2\nu}_{y-y';0;t})}$$
$$\leqslant c_1(\kappa\wedge\kappa')^{-1/2} \tag{3.13}$$

where c_1 is independent of y, $y'\in\mathbf{R}$, $t\in[0,T]$.

Recalling (3.2) and the hypotheses (2.10) on ψ_0, we have thus proven

$$\mathbf{E}(\psi^\kappa_t(x)-\psi^{\kappa'}_t(x))^2\leqslant c_1[G_t\star\psi_0(x)]^2\,(\kappa\wedge\kappa')^{-1/2}\leqslant c(\kappa\wedge\kappa')^{-1/2} \tag{3.14}$$

uniformly for $x\in\mathbf{R}$ and t in compact subsets of $(0,\infty)$.

From the $L^2(\mathscr{P})$ convergence and Lemma 3.2 we get also

$$\|\psi_t(x)\|_{L^2(\mathscr{P})}\leqslant cG_t\star\psi_0(x) \tag{3.15}$$

where c is independent of $t\in[0,T]$, $x\in\mathbf{R}$.

From the above estimate we have

$$\int_0^t ds\int_0^s ds'\int dy\, dy'\, G_{t-s}(x-y)^2\, G_{s-s'}(y-y')^2\,\mathbf{E}(\psi_{s'}(y')^2)$$
$$\leqslant c\int_0^t ds\int_0^s ds'\int dy\, dy'\, G_{t-s}(x-y)^2\, G_{s-s'}(y-y')^2$$
$$\times(s')^{-1/2}\, G_{s'}\star\psi_0(y') \tag{3.16}$$

where we used assumption (2.10) and the positivity of ψ_0.

The bound (2.11) follows then from inequality (3.16) noting that $G_t(x)^2=(4\pi t)^{-1/2}\,G_{t/2}(x)$ and using the semigroup property of the heat kernel.

We prove the other statements of the theorem.

(i) Let n be an even integer; the $L^n(\mathscr{P})$ norm of $\psi^\kappa_t(x)$ can be computed analogously to (3.1). Let $\vec{b}=(b^1,\ldots,b^n)$ be n independent Brownian bridges between y_i and x in time t. Let $L^\kappa_t(b^i-b^j):=\int_0^t ds\, C_\kappa(b^i_s-b^j_s)$; we have

$$\mathbf{E}(\psi^\kappa_t(x)^n)=\int\prod_{i=1}^n d\psi_0(y_i)\, G_t(x-y_i)\cdot\mathbf{E}^{\vec{b}}_{\vec{y},x;t}\left(\exp\sum_{i<j}L^\kappa_t(b^i-b^j)\right)\leqslant c \tag{3.17}$$

where, by Lemma 3.2 and condition (2.10), c is independent of $\kappa > 0$, $x \in \mathbf{R}$, and t in compact subsets of $(0, \infty)$.

Let $p > 1$; by the Cauchy–Schwartz inequality

$$\begin{aligned} &\|\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\|_{L^p(\mathscr{P})} \\ &\quad \leqslant \|\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\|_{L^2(\mathscr{P})}^{1/p} \cdot \|\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\|_{L^{2(p-1)}(\mathscr{P})}^{(p-1)/p} \end{aligned} \tag{3.18}$$

which converges to 0 by the $L^2(\mathscr{P})$ convergence and the uniform bound (3.17). From the $L^p(\mathscr{P})$ convergence of ψ_t^κ and Lemma 3.2 it follows that the bound (3.15) is also in $L^p(d\mathscr{P})$:

$$\|\psi_t(x)\|_{L^p(d\mathscr{P})} \leqslant c G_t \star \psi_0(x) \tag{3.19}$$

To prove the $\mathscr{P}$-a.s. convergence, we note that, by the Borel–Cantelli lemma, it is enough to show, for some $p > 1$, $\alpha > 0$, there exists $c > 0$ such that for any $\kappa > 0$, $x \in \mathbf{R}$, and t in compact subsets of $(0, \infty)$

$$\mathbf{E}\, |\psi_t^\kappa(x) - \psi_t(x)|^p \leqslant c \frac{1}{\kappa^{1+\alpha}} \tag{3.20}$$

The bound (3.20) holds for $p = 6$ with $\alpha = 1/2$. This can be proven computing, as in (3.1) and (3.2), the $L^6(\mathscr{P})$ norm of $\psi_t^\kappa(x) - \psi_t(x)$. It can be written as a sum of many (i.e., 2^6) terms; they can be associated in such a way that each of them contains the product of at least three factors of the form $\{\exp[L_t^\kappa(b^i - b^j)] - \exp[L_t(b^i - b^j)]\}$. Proceeding as in (3.13) and using again Lemmata 3.1 and 3.2, we obtain the bound (3.20). We omit the tedious algebraic details.

(ii) The bound (2.11) has already been proven. To conclude that ψ_t is the solution of the stochastic heat equation, we first verify that ψ_t^κ in (2.17) is the solution of (2.16).

From definitions (2.17) and (2.15), using the Markov property for the Brownian bridge, we have

$$\begin{aligned} &\int_0^t G_{t-s} \star \psi_s^\kappa \, dB_s^\kappa(x) \\ &\quad = \int_0^t \int d\psi_0(z)\, dy\, G_s(z-y)\, G_{t-s}(y-x)\, \mathbf{E}_{z,y;s}^b(\mathscr{E}\mathrm{xp}\{M_b^\kappa(s)\})\, dB_s^\kappa(y) \\ &\quad = \int_0^t \int d\psi_0(z)\, G_t(z-x)\, \mathbf{E}_{z,x;t}^b(\mathscr{E}\mathrm{xp}\{M_b^\kappa(s)\}\, dB_s^\kappa(b_s)) \end{aligned} \tag{3.21}$$

where we used the expression for the transition probability of the Brownian bridge.

By the definition of the stochastic curvilinear integral (2.5), $dB_s^\kappa(b_s)$ equals $dM_b^\kappa(s)$. Recalling that $\mathscr{E}\mathrm{xp}\{M_b^\kappa(s)\}$ is the Girsanov exponential (2.18), we easily compute the stochastic integral in (3.21), obtaining

$$\int d\psi_0(z)\, G_t(z-x)\, \mathbf{E}^b_{z,x;t}(\mathscr{E}\mathrm{xp}\{M_b^\kappa(t)\}-1) = \psi_t^\kappa(x) - G_t \star \psi_0(x) \quad (3.22)$$

which proves the claim.

As $\psi_t^\kappa(x)$ converges to $\psi_t(x)$ uniformly for t in compact subsets of $(0, \infty)$, $x \in \mathbf{R}$, $(t, x) \mapsto \psi_t(x)$ is $\mathscr{P}$-a.s. continuous. Thus for $t > 0$ the process ψ_t is, by construction, $\mathscr{F}_t$-adapted, continuous, and $C^0(\mathbf{R})$-valued. Since ψ_t^κ satisfies (2.16) and converges to ψ_t, once we show

$$\lim_{\kappa\to\infty} \left\| \int_0^t G_{t-s} \star \psi_s\, dB_s(x) - \int_0^t G_{t-s} \star \psi_s^\kappa\, dB_s^\kappa(x) \right\|_{L^2(\mathscr{P})} = 0 \quad (3.23)$$

we can conclude that ψ_t satisfies, $\mathscr{P}$-a.s., Eq. (2.12).

To prove (3.23) let us consider first

$$\mathbf{E}\left(\int_0^t G_{t-s} \star (\psi_s - \psi_s^\kappa)\, dB_s(x)\right)^2$$
$$= \int_0^t ds \int dy\, G_{t-s}(x-y)^2\, \mathbf{E}(\psi_s(y) - \psi_s^\kappa(y))^2 \quad (3.24)$$

Using (3.2), Lemmata 3.1 and 3.2, and (3.15), we can bound it by

$$c_1 \kappa^{-1/2} \int_0^t ds \int dy\, G_{t-s}(x-y)^2\, [G_s \star \psi_0(y)]^2$$
$$\leqslant c_2 \kappa^{-1/2} \int_0^t ds\, [s(t-s)]^{-1/2}\, G_t \star \psi_0(x) \leqslant c_3 \kappa^{-1/2} \quad (3.25)$$

where we used the hypothesis (2.10) on ψ_0.

On the other hand,

$$\mathbf{E}\left(\int_0^t G_{t-s} \star \psi_s^\kappa (dB_s - dB_s^\kappa)(x)\right)^2$$
$$= \mathbf{E} \int_0^t ds\, (G_{t-s}(x-\cdot)\psi_s^\kappa, (1-\delta_\kappa + C_\kappa - \delta_\kappa)\, G_{t-s}(x-\cdot)\psi_s^\kappa) \quad (3.26)$$

We consider just the term with $(1-\delta_\kappa)$; the other one is analogous. It can be bounded by

$$\int_0^t ds \int dy\, dy' \, [G_{t-s}(x-y)^2\, \delta_k(y-y')\, \|\psi_s^\kappa(y)\|_{L^2(\mathscr{P})}\, \|\psi_s^\kappa(y)-\psi_s^\kappa(y')\|_{L^2(\mathscr{P})}$$
$$+ G_{t-s}(x-y)\, \delta_k(y-y')\, \|\psi_s^\kappa(y)\|^2_{L^2(\mathscr{P})}$$
$$\times |G_{t-s}(x-y) - G_{t-s}(x-y')|] \tag{3.27}$$

The first line vanishes in the limit $k \to \infty$ by (3.15), the continuity of $x \to \psi_t^\kappa(x)$ and the $L^2(\mathscr{P})$ convergence (uniform in x) of $\psi_t^\kappa(x)$. For the other term we note that, by the dominated convergence theorem, we can pass to the limit inside the time integral and conclude it converges to 0. Together with (3.25) this implies (3.23).

We finally prove uniqueness in the class of processes satisfying (2.11). Since the equation is linear, it is enough to show that any solution of (2.12) with 0 initial datum is identically 0. For such a solution we have

$$\mathbf{E}(\psi_t(x)^2) = \int_0^t ds \int dy\, G_{t-s}(x-y)^2\, \mathbf{E}(\psi_s(y)^2) \tag{3.28}$$

Iterating (3.28) and using the condition (2.11), we get

$$\sup_{x\in\mathbf{R}} \mathbf{E}(\psi_t(x))^2 = \sup_{x\in\mathbf{R}} \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \int dy_1 \cdots dy_n\, G_{t-s_1}(x-y_1)^2$$
$$\times \cdots G_{s_{n-1}-s_n}(y_{n-1}-y_n)^2\, \mathbf{E}(\psi_{s_n}(y_n)^2)$$
$$\leqslant c \sup_{x\in\mathbf{R}} \int_0^t ds_1 \cdots \int_0^{s_{n-3}} ds_{n-2} \int dy_1 \cdots dy_{n-2}\, G_{t-s_1}(x-y_1)^2$$
$$\times \cdots G_{s_{n-3}-s_{n-2}}(y_{n-3}-y_{n-2})^2 \tag{3.29}$$

which converges to 0 as $n \to \infty$.

(iii) We first show the Hölder continuity of $t \mapsto \psi_t(x)$. Thanks to the bound (3.15), it can be easily verified that for any $t>0$, $\delta>0$ there exists c such that for any $h>0$, $x\in\mathbf{R}$,

$$\left\| \int_0^{t+h} G_{t+h-s} \star \psi_s\, dB_s(x) - \int_0^t G_{t-s} \star \psi_s\, dB_s(x) \right\|_{L^2(\mathscr{P})} \leqslant ch^{1/4-\delta} \tag{3.30}$$

By an application of the Burkholder–Davis–Gundy inequality (see, e.g., ref. 17, Section 4.2) and using (3.19) instead of (3.15), we can prove the same bound also in $L^p(\mathscr{P})$. We omit the details; see, however, ref. 2 for an analogous estimate. Using Eq. (2.12) and the Kolmogorov criterium, the $\mathscr{P}$-a.s. Hölder continuity follows.

The following bound is proven with the same argument as above. For any $p > 1$, $t > 0$, $\delta > 0$ there exists c such that for any $h > 0$, $x \in \mathbf{R}$,

$$\left\| \int_0^t G_{t-s} \star \psi_s \, dB_s(x+h) - \int_0^t G_{t-s} \star \psi_s \, dB_s(x) \right\|_{L^p(\mathscr{P})} \leqslant c h^{1/2-\delta} \tag{3.31}$$

This implies that $x \mapsto \psi_t(x)$ is $\mathscr{P}$-a.s. Hölder continuous with exponent $\alpha < 1/2$.

(iv) Let first note that a comparison principle holds. Let ψ_t^i, $i = 1, 2$, be the solution with initial datum ψ_0^i; from the linearity of the equation and the Feynman–Kac formula we have that $\psi_0^1 \leqslant \psi_0^2$ (as measures) implies

$$\forall (t, x) \in (0, \infty) \times \mathbf{R}, \qquad \psi_t^1(x) \leqslant \psi_t^2(x) \qquad \mathscr{P}\text{-a.s.} \tag{3.32}$$

For initial conditions which are absolutely continuous with respect to Lebesgue and whose density is continuous and with compact support the strict inequality $\psi_t(x) > 0$ is proven in Mueller[14] up to an explosion time which is infinite by our results.

Using Mueller's result and the comparison relation (3.32), we prove statement (iv) for initial data with continuous density with respect to Lebesgue. General initial data are then reduced to this case thanks to the Markov property and the fact that they became continuous in space in an arbitrary small time. ▌

4. STATISTICAL PROPERTIES

In this section we prove the explicit formulas for the correlation functions and the intermittent behavior of the solution we constructed. Proposition 2.3, which allows us to express the moments of ψ_t in terms of local times, is a straightforward consequence of the machinery already developed. Corollaries 2.4 and 2.5 follow then from known results on the distribution of local times. Finally, Theorem 2.6 exploits elementary properties of the local times: the nth moment is computed, reducing it to the evaluation of an exponential moment for the local time of a single Brownian motion.

Proof of Proposition 2.3. For any integer n, by Theorem 2.2(i),

$$\mathbf{E}(\psi_t(x)^n) = \lim_{\kappa \to \infty} \mathbf{E}(\psi_t^\kappa(x)^n) \tag{4.1}$$

Introducing n independent copies of b and computing the expectation with respect to $\mathscr{P}$, we have

$$\mathbf{E}(\psi_t^\kappa(x)^n) = \int \prod_{i=1}^n d\psi_0(y_i)\, G_t(x-y_i) \cdot \mathbf{E}^{\bar{b}}_{\bar{y},x;t}\left(\exp\left\{\sum_{i<j}\int_0^t ds\, C_\kappa(b_s^i - b_s^j)\right\}\right) \tag{4.2}$$

By Lemmata 3.1 and 3.2, the right-hand side of (4.2) converges to

$$\int \prod_{i=1}^n d\psi_0(y_i)\, G_t(x-y_i) \cdot \mathbf{E}^{\bar{b}}_{\bar{y},x;t}\left(\exp \sum_{i<j} L_t(b^i - b^j)\right) \tag{4.3}$$

which proves (2.22). The proof of (2.23) is analogous. ∎

Proof of Corollaries 2.4 and 2.5. If ψ_0 is the Lebesgue measure, we can express the correlation functions in terms of local times of Brownian motions. Let β_s, $s \geqslant 0$, be the Brownian motion, with diffusion coefficient ν, starting from x; denote by dP_x^β its law. Realizing the Brownian bridge as conditional Brownian motion, we have

$$dP_x^{\beta,\nu} = \int dy\, G_t(x-y)\, dP^{b,\nu}_{y,x;t} \tag{4.4}$$

By (2.23) we can thus express the correlation function as

$$\begin{aligned}\mathbf{E}(\psi_t(x)\,\psi_t(x')) &= \mathbf{E}^{\beta,2\nu}_{x-x'}(\exp L_t(\beta)) = \mathbf{E}_0^{\beta,2\nu}(\exp L_t^{x-x'}(\beta)) \\ &= \mathbf{E}_0^{\beta,1}(\exp[(2\nu)^{-1/2} L_t^{(x-x')/(2\nu)^{1/2}}(\beta)])\end{aligned} \tag{4.5}$$

where the last identity is obtained by scaling. Let $\xi := (x-x')/(2\nu)^{1/2}$, introduce the stopping time $T_\xi := \inf\{s\colon \beta_s = \xi\}$, and denote by $P_\xi(ds)$ its law. By the strong Markov property and the additivity of the local time we have

$$\begin{aligned}\mathbf{E}_0^{\beta,1}(e^{(2\nu)^{-1/2} L_t^\xi(\beta)}) &= \int_0^t P_\xi(ds)\, \mathbf{E}_0^\beta(e^{(2\nu)^{-1/2} L_{t-s}(\beta)}) \\ &= 2\int_0^t ds \frac{|\xi|}{(2\pi s^3)^{1/2}}\, e^{-\xi^2/2s} \int_0^\infty dy\, G_{t-s}(y)\, e^{y(2\nu)^{-1/2}}\end{aligned} \tag{4.6}$$

where we used the explicit expression for $P_\xi(ds)$ (ref. 17, Chapter III, 3.7) and for the law of $L_t(\beta)$, ref. 17, Chapter VI, 2.2. Equation (2.25) is just a convenient rewriting of (4.6).

Corollary 2.5 is proven following the same steps. When $\psi_0(dx) = \delta_0(dx)$, from (2.23) we have

$$\mathbf{E}(\psi_t(x)\,\psi_t(x')) = \frac{1}{2\pi\nu t}\, e^{-[x^2 + (x')^2]/2\nu t}\, \mathbf{E}^{b,1}_{0,\,(x-x')/(2\nu t)^{1/2};\,1}(e^{(t/2\nu)\,L_1(b)}) \tag{4.7}$$

On the other hand, by the time-reversal property of the Brownian bridge,

$$\mathbf{E}^{b,1}_{0,a;1}(e^{\lambda L_1(b)}) = \mathbf{E}^{b,1}_{a,0;1}(e^{\lambda L_1^a(b)}) = \int_0^1 \tilde{P}_a(ds)\, \mathbf{E}^{b,1}_{0,0;1-s}(e^{\lambda L_1(b)})$$

$$= \int_0^1 \tilde{P}_a(ds)\, \mathbf{E}^{b,1}_{0,0;1}(e^{\lambda(1-s)^{1/2} L_1(b)}) \tag{4.8}$$

where $\tilde{P}_a(ds)$ is the law of the stopping time $T_a := \inf\{t: b_t = a\}$ for the Brownian bridge from 0 to a in time 1; realizing it as a conditional Brownian motion, it can be verified that

$$\tilde{P}_a(ds) = \frac{|a|}{[2\pi s^3(1-s)]^{1/2}}\, e^{-(a^2/2)(1-s)/s}\, ds \tag{4.9}$$

The formula (2.26) follows then from (4.7)–(4.9) and the following result (see ref. 17, Chapter XII, 3.8). If b is a Brownian bridge (with diffusion coefficient 1) from 0 to 0 in time 1, the local time $L_1(b)$ has the same law of $(2\gamma)^{1/2}$, where γ is an exponential random variable of mean 1. ∎

Proof of Theorem 2.6. As in the proof of Corollary 2.4, we express the moments of $\psi_t(x)$ in terms of the local times of independent Brownian motions. Let $\vec{\beta} := (\beta^1, \dots, \beta^n)$ be n independent Brownian motions with diffusion coefficient ν; from Proposition 2.3 we have

$$\mathbf{E}(\psi_t(x)^n) = \mathbf{E}^{\vec{\beta},\nu}_x\left(\exp \sum_{i<j} L_t(\beta^i - \beta^j)\right) \tag{4.10}$$

where we used (4.4).

By the Tanaka formula (2.20) we have

$$\sum_{i<j} |\beta^i_t - \beta^j_t| = 2\nu w_t + 2\nu \sum_{i<j} L_t(\beta^i - \beta^j) \tag{4.11}$$

where

$$w_t := \frac{1}{2\nu} \sum_{i<j} \int_0^t \operatorname{sgn}(\beta^i_s - \beta^j_s)\, d(\beta^i_s - \beta^j_s) \tag{4.12}$$

It can be rewritten as

$$w_t = \frac{1}{2\nu} \sum_{i=1}^n \int_0^t \left(\sum_{j\neq i} \operatorname{sgn}(\beta^i_s - \beta^j_s)\right) d\beta^i_s \tag{4.13}$$

from which we get

$$\langle w, w\rangle_t = \frac{t}{4\nu}\sum_{i=1}^{n}[n-1-2(i-1)]^2 = \frac{t}{4\nu}\frac{n(n^2-1)}{3} \tag{4.14}$$

By the Levy characterization theorem w is thus, in law, a Brownian motion starting from 0 and with diffusion coefficient $n(n^2-1)/12\nu$.

Using a deterministic procedure in (4.11), the Skorohod lemma (see ref. 17, Chapter VI, 2.1), we have

$$\sum_{i<j} L_t(\beta^i - \beta^j) = \sup_{s\leqslant t}(-w_s) \tag{4.15}$$

Recalling (4.10), we have thus proven

$$\begin{aligned}\mathbf{E}(\psi_t(x)^n) &= \mathbf{E}_x^{\bar{\beta},\nu}(\exp\{\sup_{s\leqslant t}(-w_s)\}) \\ &= \mathbf{E}_0^{\beta,1}\left(\exp\left\{\left[\frac{n(n^2-1)}{12\nu}\right]^{1/2}\sup_{s\leqslant t}(-\beta_s)\right\}\right)\end{aligned} \tag{4.16}$$

from which (2.39) follows by the reflection principle: if β is a Brownian motion starting from 0, then $\sup_{s\leqslant t}\beta_s$ has the same law as $|\beta_t|$ (see, e.g., ref. 17, Chapter III, 3.7). ∎

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